9. Newton's Laws of Dynamics

9.1. Galileo discovered the principle of inertia: objects have a 'reluctance' to change their state of motion.

- Newton's laws quantify and expand on the above.
  - The first law is a restatement of Galileo's principle
  - The second law describes how the velocity changes as a result of forces
  - The third law describes a characteristic of forces

Mathematically, the second law states that

\[ F = \frac{d}{dt}(mv) = m \frac{dv}{dt} = ma \]

- Mass is assumed to be constant
- \( v \) is a vector quantity

This also applies to circular motion. In 7.4, we showed that the vertical distance an object 'falls' while undergoing circular motion is found by

\[
S = \frac{x^2 - (vt)^2}{2R} = \frac{1}{2} \left( \frac{v^2}{R} \right) t^2.
\]

For \( u = 0 \), this is equivalent to one of the satellite equations with \( a = \frac{v^2}{R} \).

9.2. Splitting motion into components

- We can completely specify the magnitude and direction of an action by considering the \( x, y, z \) components.
  - E.g. for velocity:
    \[
    v_x = \frac{dx}{dt}, \quad v_y = \frac{dy}{dt}, \quad v_z = \frac{dz}{dt}
    \]
    the speed is then \( |v| = \sqrt{v_x^2 + v_y^2 + v_z^2} \).

9.3. We can do the same thing with forces:

\( F_x = ma_x = F \cos(x, \theta) \) where \( F = \sqrt{F_x^2 + F_y^2 + F_z^2} \)

\( F_y = ma_y = F \cos(y, \theta) \) and \( (x, \theta) \) is the angle between \( x \) and \( F \).
9.4 Laws for forces

- Newton discovered that \( F = \frac{GMm}{r^2} = mg \), where \( g \) is the acceleration of gravity.
- By \( \text{N}2 \), we have \( mg = ma \), so \( g = a \).
- Then, \( \dot{v}_z = v_0 + gt \Rightarrow z = z_0 + v_0 t + \frac{1}{2} gt^2 \).

- A spring obeys the equation \( F = -kx \).
- By \( \text{N}2 \), we have \(-kx = m \frac{dv_x}{dt} \).
- Choosing suitable units: \( \frac{dv_x}{dt} = -cx \).

9.5 The time-evolution of a system

- At time \( t \), an object may have position \( x \) and velocity \( v \).
- We then want to know what happens at \( t + \varepsilon \).
- For small \( \varepsilon \), we can say
  \[ x(t + \varepsilon) = x(t) + \varepsilon v(t). \]

- Likewise:
  \[ v(t + \varepsilon) = v(t) + \varepsilon a(t). \]
- The dynamics (not kinematics) of the system allow for \( u \) to proceed. For example, in the case of a spring, we substitute \( a(t) = -cx(t) \).

To summarise:
- initial \( v \) and \( x \) \( \rightarrow \) new \( x \) \( \rightarrow \) new \( a \)...
- initial acceleration \( \downarrow \)
- new \( v \) \( \uparrow \)
9.6 The numerical solution for the spring system

Let \( x(0) = 1 \), \( v(0) = 0 \), \( \varepsilon = 0.1 \)

\[
\begin{align*}
x(0.1) &= x(0) + 0.1v(0) = 1 \\
v(0.1) &= v(0) + 0.1a(0) = v(0) - 0.1x(0) = -0.1
\end{align*}
\]

THEN \( x(0.2) = x(0.1) + 0.1v(0.1) = 0.99 \)

\( v(0.2) = v(0.1) - 0.1x(0.1) = -0.2 \)

and so on.

Above, we used the velocity value at the lower boundary \([e.g.\ v(0)]\), but we can improve by using the midpoints.

\[
\begin{align*}
x(t + \varepsilon) &= x(t) + \varepsilon v(t + \frac{\varepsilon}{2}) \\
v(t + \frac{\varepsilon}{2}) &= v(t - \frac{\varepsilon}{2}) + \varepsilon a(t) \\
a(t) &= -\varepsilon c(t)
\end{align*}
\]

However, note that we will need a special equation to start the \( v \)-values off:

\[
v(\frac{\varepsilon}{2}) = v(0) + \frac{\varepsilon}{2} a(0).
\]

We can now fill up a table and plot a graph of motion.

9.7 Planetary motion

We set up a coordinate system, with the sun (stationary) at \((0, 0)\) and the planet at \((x, y)\). We then split the gravitational force into \( F_x \) and \( F_y \)

Because of similar triangles,

\[
F_x = -\frac{x}{r} \Rightarrow F_x = -\frac{F}{r}
\]

\[
F_x = -\frac{GMm_x}{r^3}
\]

(same can be done for \( F_y \))
Then, with the right units, we have:
\[ a_x = -\frac{x}{r^3} \]
\[ a_y = -\frac{y}{r^3} \]
\[ r = \sqrt{x^2 + y^2} \]

Choosing initial values:
\[ x(0) = 0.5 \quad y(0) = 0 \quad \varepsilon = 0.1 \]
\[ v_x(0) = 0 \quad v_y(0) = 1.63 \]

\[ \Rightarrow \quad r(0) = 0.5 \]
\[ a_x(0) = -4 \quad a_y(0) = 0 \]
\[ \text{then} \quad v_x(0.05) = v_x(0) + 0.05a_x(0) = 0 - 4 \times 0.05 = -0.2 \]
\[ v_y(0.05) = v_y(0) + 0.05a_y(0) = 1.63 + 0 \times 0.05 = 1.63 \]

\[ \Rightarrow \text{calculate } x(0.1) \text{ and } y(0.1) \Rightarrow \text{calculate } r \Rightarrow \text{calculate } a(0.1) \]

Repeating this process gives an elliptical shape.

We can extend this to include perturbations. If we are planet \( i \), and there are \( N \) other planets:
\[ m_i a_i x = \sum_{j=1}^{N} \frac{-Gm_i m_j (x - x_j)}{r_{ij}^3} \]

and likewise for \( a_i y \) and \( a_i z \).